

NEW PARAMETRIZATION METHODS FOR GENERATING ADOMIAN POLYNOMIALS

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ABSTRACT. In this paper, we develop a general parametrization technique for calculating Adomian polynomials for several nonlinear operators. Some important properties of Adomian polynomials are also discussed and illustrated with examples. We develop two new parametrization methods for calculating Adomian polynomials, which utilize the orthogonality of functions $\{e^{inx}, n \in \mathbb{Z}\}$. These methods require minimum computation, are easy to implement and generate Adomian polynomials in fewer steps. These methods are then extended to multivariable case also. Examples of different forms of nonlinearity, which include Navier Stokes equation and time fractional nonlinear Schrödinger equation, are considered and explicit expression for the n -th order Adomian polynomials are obtained.

1. INTRODUCTION

The Adomian decomposition method (ADM) [1, 2, 3] provides an analytical approximate solution for nonlinear functional equation in terms of a rapidly converging series, without linearization, perturbation or discretization. Consider a functional equation

$$u = f + L(u) + N(u), \quad (1.1)$$

where L and N are respectively, linear and nonlinear operators from a Hilbert space H into H and f is a known function in H . In ADM, the solution $u(x, t)$ of (1.1) is decomposed in the form of an infinite series given by

$$u(x, t) = \sum_{k=0}^{\infty} u_k(x, t). \quad (1.2)$$

Further, the nonlinear function $N(u)$ is assumed to admit the representation

$$N(u) = \sum_{k=0}^{\infty} A_k(u_0, u_1, \dots, u_k), \quad (1.3)$$

where A_k 's are called k -th order Adomian polynomials. In the linear case $N(u) = u$, A_k simply reduces to u_k . Adomian's method is simple in principle, but involves tedious calculations of Adomian polynomials. Adomian [1] gave a method for determining these Adomian polynomials, by parametrizing $u(x, t)$ as

$$u_{\lambda}(x, t) = \sum_{k=0}^{\infty} u_k(x, t) \lambda^k \quad (1.4)$$

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and assuming $N(u_\lambda)$ to be analytic in λ , which decomposes as

$$N(u_\lambda) = \sum_{k=0}^{\infty} A_k(u_0, u_1, \dots, u_k) \lambda^k. \quad (1.5)$$

Hence, the Adomian polynomials A_n are given by

$$A_n(u_0, u_1, \dots, u_n) = \frac{1}{n!} \left. \frac{\partial^n N(u_\lambda)}{\partial \lambda^n} \right|_{\lambda=0}, \quad \forall n \in \mathbb{N}_0, \quad (1.6)$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and \mathbb{N} denotes the set of positive integers. Rach [2, 3, 12] suggested the following formula for determining Adomian polynomials:

$$\begin{aligned} A_0(u_0) &= N(u_0), \\ A_n(u_0, u_1, \dots, u_n) &= \sum_{k=1}^n C(k, n) N^{(k)}(u_0), \quad \forall n \in \mathbb{N}, \end{aligned} \quad (1.7)$$

where $C(k, n)$ is the product (or sum of products) of k components of $u(x, t)$ whose subscripts sum to n divided by the factorial of the number of repeated subscripts, that is,

$$C(k, n) = \sum_{\substack{\sum_{j=1}^n j k_j = n \\ \sum_{j=1}^n k_j = k, \quad k_j \in \mathbb{N}_0}} \prod_{j=1}^n \frac{u_j^{k_j}}{k_j!}. \quad (1.8)$$

Wazwaz [15] suggested a new algorithm in which after separating $A_0 = N(u_0)$ from other terms of the Taylor series expansion of the nonlinear function $N(u)$, we collect all terms of the expansion obtained such that the sum of the subscripts of the components of $u(x, t)$ in each term is same. The limitations of this algorithm is that it is difficult to keep track of the terms after some time. Zhu *et al.* [16] suggested another useful method, but it also involves tedious calculations of n -th derivative to obtain A_n . Biazar and Shafiof [7] proposed a recursive method to calculate Adomian polynomials, in which only one time differentiation is required. However, the disadvantage is that we do not have explicit form for A_n 's.

In this paper, we develop a general parametrization technique for calculating Adomian polynomials and discuss some of their important properties. Indeed, we develop two new simple methods to generate Adomian polynomials using the orthogonality of functions $\{e^{inx}, n \in \mathbb{Z}\}$. The first method determines these polynomials explicitly, whereas the second method generates them recursively. The newly developed techniques are more viable, require less computation and generate Adomian polynomials in a fewer steps. Both the methods are extended to the case of several variables. Different forms of nonlinearity are discussed as applications of our methods.

2. ADOMIAN POLYNOMIALS AND PARAMETRIZATION METHODS

We assume the following hypotheses [8]:

- $H1$: The series solution $u = \sum_{k=0}^{\infty} u_k$ of (1.1) is absolutely convergent and,
- $H2$: The nonlinear function $N(u)$ is developable into an entire series with a convergence radius equal to infinity, that is,

$$N(u) = \sum_{k=0}^{\infty} N^{(k)}(0) \frac{u^k}{k!}, \quad |u| < \infty. \quad (2.1)$$

The second assumption is almost always satisfied in concrete physical problems. By $H1$ and $H2$, we have Adomian series as a generalization of Taylor series [8],

$$N(u) = \sum_{k=0}^{\infty} A_k(u_0, u_1, \dots, u_k) = \sum_{k=0}^{\infty} N^{(k)}(u_0) \frac{(u - u_0)^k}{k!}. \quad (2.2)$$

Note that (2.2) is a rearrangement of an absolutely convergent series (2.1). We look at a more general form of parametrization than the one given in (1.4). That is, we consider

$$u_\lambda(x, t) = \sum_{k=0}^{\infty} u_k(x, t) f^k(\lambda), \quad (2.3)$$

where λ is a real parameter and f is any real or complex valued function with $|f| < 1$. Note that for such parametrization function f , series (2.3) is also absolutely convergent.

Remark 2.1. When chosen parametrization function f in (2.3) is complex valued and the complex conjugate $\bar{u}(x, t)$ of $u(x, t)$ appears in nonlinear function $N(u)$, then $\bar{u}(x, t)$ is parametrized as

$$\bar{u}_\lambda(x, t) = \sum_{k=0}^{\infty} \bar{u}_k(x, t) f^k(\lambda). \quad (2.4)$$

Now substituting (2.3) in (2.2) we have

$$N(u_\lambda) = \sum_{k=0}^{\infty} N^{(k)}(u_0) \frac{\left(\sum_{j=1}^{\infty} u_j(x, t) f^j(\lambda) \right)^k}{k!}. \quad (2.5)$$

Since $\sum_{j=1}^{\infty} u_j(x, t) f^j(\lambda)$ is absolutely convergent, we can rearrange $N(u_\lambda)$ in a series form of the type, $\sum_{k=0}^{\infty} A_k f^k(\lambda)$. Therefore, from (2.5) we collect the coefficients A_k of $f^k(\lambda)$, which leads to Adomian polynomials. That is,

$$\begin{aligned} N(u_\lambda) &= N(u_0) + N^{(1)}(u_0) (u_1 f(\lambda) + u_2 f^2(\lambda) + \dots) \\ &\quad + \frac{N^{(2)}(u_0)}{2!} (u_1 f(\lambda) + u_2 f^2(\lambda) + \dots)^2 \\ &\quad + \frac{N^{(3)}(u_0)}{3!} (u_1 f(\lambda) + u_2 f^2(\lambda) + \dots)^3 + \dots \\ &= N(u_0) + N^{(1)}(u_0) u_1 f(\lambda) + \left(N^{(1)}(u_0) u_2 + N^{(2)}(u_0) \frac{u_1^2}{2!} \right) f^2(\lambda) \\ &\quad + \left(N^{(1)}(u_0) u_3 + N^{(2)}(u_0) u_1 u_2 + N^{(3)}(u_0) \frac{u_1^3}{3!} \right) f^3(\lambda) + \dots \\ &= \sum_{k=0}^{\infty} A_k(u_0, u_1, \dots, u_k) f^k(\lambda). \end{aligned} \quad (2.6)$$

Also note that A_k 's are polynomials in u_0, u_1, \dots, u_k only. For a suitable choice of f , we possibly can develop a convenient method to determine these Adomian polynomials. One such method was given by Adomian himself where he choose $f(\lambda) = \lambda$ and then taking n -th derivative on both sides of (2.6) obtained (1.6). In Section 4, we choose $f(\lambda) = e^{i\lambda}$ and develop two new methods to determine Adomian polynomials.

3. SOME PROPERTIES OF ADOMIAN POLYNOMIALS

In this section, we discuss some important properties of Adomian polynomials, which are very useful and in many cases we can get Adomian polynomials for certain nonlinear operators without explicit calculations. As far as calculations of Adomian polynomials are concerned, formal power series can be used efficiently. Formal power series are purely algebraic objects, and can be defined without the notion of convergence. In order to obtain Adomian polynomials, we utilize some well known operations on formal power series. Let f and g be formal power series in x with $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$. Then,

$$\frac{g(x)}{f(x)} = \sum_{k=0}^{\infty} c_k x^k, \quad c_0 = \frac{b_0}{a_0}, \quad c_k = \frac{1}{a_0} \left(b_k - \sum_{j=1}^k a_j c_{k-j} \right), \quad (3.1)$$

and

$$f^n(x) = \sum_{k=0}^{\infty} c_k x^k, \quad c_0 = a_0^n, \quad c_k = \frac{1}{k a_0} \sum_{j=1}^k (j n - k + j) a_j c_{k-j}, \quad (3.2)$$

provided a_0 is invertible in the ring of scalars.

Theorem 3.1. Let $A_{1n}, A_{2n}, \dots, A_{mn}$, $n \geq 1$, be the Adomian polynomials corresponding to nonlinear operators N_1, N_2, \dots, N_m , respectively. Then the Adomian polynomials of

- (i) $N(u) = \sum_{k=1}^m \alpha_k N_k(u)$ are given by $A_n = \sum_{k=1}^m \alpha_k A_{k_n} \quad \forall n \in \mathbb{N}_0$, where the α_k are scalars.
- (ii) $N(u) = \prod_{k=1}^m N_k(u)$ are given by

$$A_n = \sum_{\substack{\sum_{j=1}^m k_j = n \\ k_j \in \mathbb{N}_0}} \prod_{j=1}^m A_{j k_j}, \quad \forall n \in \mathbb{N}_0. \quad (3.3)$$

In particular, Adomian polynomials of $N(u) = N_1(u)N_2(u)$ are

$$A_n = \sum_{k=0}^n A_{1_k} A_{2_{n-k}}.$$

- (iii) $N(u) = \frac{N_1(u)}{N_2(u)}$ are given by $A_0 = \frac{A_{1_0}}{A_{2_0}}$ and

$$A_n = \frac{1}{A_{2_0}} \left(A_{1_n} - \sum_{k=1}^n A_{2_k} A_{n-k} \right), \quad \forall n \in \mathbb{N}. \quad (3.4)$$

- (iv) $N(u) = N_1^p(u)$ for any $p \in \mathbb{N}$ are given by $A_0 = A_{1_0}^p$ and

$$A_n = \frac{1}{n A_{1_0}} \sum_{k=1}^n (k p - n + k) A_{1_k} A_{n-k}, \quad \forall n \in \mathbb{N}. \quad (3.5)$$

- (v) $N(u) = N_1(N_2(u))$ are given by $A_0 = N_1(A_{2_0})$ and

$$A_n = \sum_{\substack{\sum_{j=1}^n j k_j = n \\ k_j \in \mathbb{N}_0}} N_1^{(\sum_{j=1}^n k_j)}(A_{2_0}) \prod_{j=1}^n \frac{A_{2_j}^{k_j}}{k_j!}, \quad \forall n \in \mathbb{N}. \quad (3.6)$$

Proof. (i) Directly follows from (1.6).

(ii) Note that Leibniz rule [11] for higher derivatives of product of m functions is given by

$$\frac{d^n}{dt^n} (f_1(t)f_2(t) \dots f_m(t)) = \sum_{\substack{\sum_{j=1}^m k_j = n \\ k_j \in \mathbb{N}_0}} n! \prod_{j=1}^m \frac{f_j^{(k_j)}(t)}{k_j!}. \quad (3.7)$$

Using (1.6) and (3.7), the Adomian polynomials are

$$\begin{aligned} A_n(u_0, u_1, \dots, u_n) &= \frac{1}{n!} \frac{\partial^n N_1(u_\lambda) N_2(u_\lambda) \dots N_m(u_\lambda)}{\partial \lambda^n} \Big|_{\lambda=0} \\ &= \frac{1}{n!} \sum_{\substack{\sum_{j=1}^m k_j = n \\ k_j \in \mathbb{N}_0}} n! \prod_{j=1}^m \frac{1}{k_j!} \frac{\partial^{k_j} N_j(u_\lambda)}{\partial \lambda^{k_j}} \Big|_{\lambda=0} \\ &= \sum_{\substack{\sum_{j=1}^m k_j = n \\ k_j \in \mathbb{N}_0}} \prod_{j=1}^m A_{j k_j}, \quad \forall n \in \mathbb{N}_0. \end{aligned}$$

(iii) Follows directly from (1.5) and (3.1) whereas (iv) follows from (1.5) and (3.2).

(v) Adomian [4] proposed an algorithm for the Adomian polynomials of composite nonlinearity. We hereby give an explicit formula for the same by using Faà di Bruno's formula [11] for generalized chain rule for higher derivatives of composition of two functions given by

$$\frac{d^n}{dt^n} g(f(t)) = \sum_{\substack{\sum_{j=1}^n j k_j = n \\ k_j \in \mathbb{N}_0}} n! g^{(\sum_{j=1}^n k_j)}(f(t)) \prod_{j=1}^n \frac{1}{k_j!} \left(\frac{f^{(j)}(t)}{j!} \right)^{k_j}, \quad \forall n \in \mathbb{N}. \quad (3.8)$$

Hence, from (1.6) and using (3.8), we have

$$\begin{aligned} A_n(u_0, u_1, \dots, u_n) &= \frac{1}{n!} \frac{\partial^n N_1(N_2(u_\lambda))}{\partial \lambda^n} \Big|_{\lambda=0} \\ &= \sum_{\substack{\sum_{j=1}^n j k_j = n \\ k_j \in \mathbb{N}_0}} N_1^{(\sum_{j=1}^n k_j)}(N_2(u_\lambda)) \prod_{j=1}^n \frac{1}{k_j!} \left(\frac{1}{j!} \frac{\partial^j N_2(u_\lambda)}{\partial \lambda^j} \right)^{k_j} \Big|_{\lambda=0} \\ &= \sum_{\substack{\sum_{j=1}^n j k_j = n \\ k_j \in \mathbb{N}_0}} N_1^{(\sum_{j=1}^n k_j)}(A_{20}) \prod_{j=1}^n \frac{A_{2j}^{k_j}}{k_j!}, \quad \forall n \in \mathbb{N}. \end{aligned}$$

□

Remark 3.1. Rach formula (1.7) is a particular case of (3.6) for composed function $N(u_\lambda)$.

4. TWO NEW METHODS TO CALCULATE ADOMIAN POLYNOMIALS

In this section, we give two new methods to calculate Adomian polynomials. The basic idea is to avoid the tedious calculations of higher derivatives involved in prevalent methods.

Consider the set of orthogonal functions $\{e^{inx}, n \in \mathbb{Z}\}$, which indeed forms a basis for the Hilbert space $L^2[-\pi, \pi]$ with inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) \overline{g(x)} dx.$$

Specifically, we use the fact

$$\langle e^{in\lambda}, e^{im\lambda} \rangle = \int_{-\pi}^{\pi} e^{in\lambda} e^{-im\lambda} d\lambda = \begin{cases} 0 & \text{if } m \neq n, \\ 2\pi & \text{if } m = n. \end{cases} \quad (4.1)$$

We choose $f(\lambda) = e^{i\lambda}$ in (2.3), to obtain

$$u_\lambda = \sum_{k=0}^{\infty} u_k e^{ik\lambda} \quad (4.2)$$

and from Remark 2.1, its complex conjugate, $\bar{u}(x, t)$ is parametrized as $\bar{u}_\lambda = \sum_{k=0}^{\infty} \bar{u}_k e^{ik\lambda}$.

Remark 4.1. Note that u_λ in (4.2), as a function of λ , is a series of periodic functions each of period 2π and therefore $N(u_\lambda)$ is also 2π -periodic. The absolute convergence of $u_\lambda(x, t) = \sum_{k=0}^{\infty} u_k e^{ik\lambda}$ and $N(u_\lambda)$ follow from hypotheses $H1$ and $H2$. Also, for parametrization (4.2), Adomian polynomials for the nonlinear function $N(u)$ turn out to be the Fourier coefficients of the periodic function $N(u_\lambda)$.

Theorem 4.1. Let $u_\lambda = \sum_{k=0}^{\infty} u_k e^{ik\lambda}$ be a parametrized representation of $u(x, t)$, where λ is a real parameter and N be the nonlinear function defined in (1.1). Then,

$$\int_{-\pi}^{\pi} N(u_\lambda) e^{-in\lambda} d\lambda = \int_{-\pi}^{\pi} N\left(\sum_{k=0}^n u_k e^{ik\lambda}\right) e^{-in\lambda} d\lambda, \quad \forall n \in \mathbb{N}_0. \quad (4.3)$$

Proof. From the first assumption $H1$, $\sum_{k=0}^{\infty} |u_k| = M < \infty$. Therefore, we have from (2.2),

$$|N(u_\lambda)| = \left| \frac{N^{(k)}(u_0)}{k!} \left(\sum_{j=1}^{\infty} u_j e^{ij\lambda} \right)^k \right| \leq \left| \frac{N^{(k)}(u_0)}{k!} \right| \left(\sum_{j=1}^{\infty} |u_j| \right)^k = \left| \frac{N^{(k)}(u_0)}{k!} \right| M^k,$$

where $M = \sum_{j=1}^{\infty} |u_j|$. Since (2.2) is an absolutely convergent series with infinite radius of convergence, $\sum_{k=0}^{\infty} \left| \frac{N^{(k)}(u_0)}{k!} \right| M^k$ converges. By Weierstrass M-test, the series

$$\sum_{k=0}^{\infty} \frac{N^{(k)}(u_0)}{k!} \left(\sum_{j=1}^{\infty} u_j e^{ij\lambda} \right)^k$$

converges uniformly. Hence, using (2.2), we get for $n \in \mathbb{N}_0$

$$\begin{aligned} \int_{-\pi}^{\pi} N(u_\lambda) e^{-in\lambda} d\lambda &= \int_{-\pi}^{\pi} \sum_{k=0}^{\infty} \frac{N^{(k)}(u_0)}{k!} \left(\sum_{j=1}^n u_j e^{ij\lambda} + \sum_{j=n+1}^{\infty} u_j e^{ij\lambda} \right)^k e^{-in\lambda} d\lambda \\ &= \int_{-\pi}^{\pi} \lim_{m \rightarrow \infty} \sum_{k=0}^m \frac{N^{(k)}(u_0)}{k!} \left(\sum_{j=1}^n u_j e^{ij\lambda} + \sum_{j=n+1}^{\infty} u_j e^{ij\lambda} \right)^k e^{-in\lambda} d\lambda \\ &= \lim_{m \rightarrow \infty} \int_{-\pi}^{\pi} \sum_{k=0}^m \frac{N^{(k)}(u_0)}{k!} \left(\sum_{j=1}^n u_j e^{ij\lambda} + \sum_{j=n+1}^{\infty} u_j e^{ij\lambda} \right)^k e^{-in\lambda} d\lambda \end{aligned}$$

$$\begin{aligned}
&= \lim_{m \rightarrow \infty} \sum_{k=0}^m \int_{-\pi}^{\pi} \frac{N^{(k)}(u_0)}{k!} \left(\sum_{j=1}^n u_j e^{ij\lambda} \right)^k e^{-in\lambda} d\lambda \quad (\text{using (4.1)}) \\
&= \lim_{m \rightarrow \infty} \int_{-\pi}^{\pi} \sum_{k=0}^m \frac{N^{(k)}(u_0)}{k!} \left(\sum_{j=1}^n u_j e^{ij\lambda} \right)^k e^{-in\lambda} d\lambda \\
&= \int_{-\pi}^{\pi} \sum_{k=0}^{\infty} \frac{N^{(k)}(u_0)}{k!} \left(\sum_{j=0}^n u_j e^{ij\lambda} - u_0 \right)^k e^{-in\lambda} d\lambda \\
&= \int_{-\pi}^{\pi} N \left(\sum_{k=0}^n u_k e^{ik\lambda} \right) e^{-in\lambda} d\lambda,
\end{aligned}$$

where the last step follows from (2.2). This completes the proof. \square

Using Theorem 4.1, we propose two new methods to calculate Adomian polynomials.

First Method. Let $u_\lambda = \sum_{k=0}^{\infty} u_k e^{ik\lambda}$, and $N(u_\lambda) = \sum_{k=0}^{\infty} A_k e^{ik\lambda}$, where A_k 's are Adomian polynomials. Then

$$\int_{-\pi}^{\pi} N \left(\sum_{k=0}^{\infty} u_k e^{ik\lambda} \right) e^{-in\lambda} d\lambda = \int_{-\pi}^{\pi} \sum_{k=0}^{\infty} A_k e^{ik\lambda} e^{-in\lambda} d\lambda = 2\pi A_n. \quad (4.4)$$

The last equality in (4.4) follows due to the uniform convergence of $\sum_{k=0}^{\infty} A_k e^{i(k-n)\lambda}$ and by using (4.1). Hence,

$$A_n(u_0, u_1, \dots, u_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} N \left(\sum_{k=0}^{\infty} u_k e^{ik\lambda} \right) e^{-in\lambda} d\lambda. \quad (4.5)$$

Applying Theorem 4.1,

$$A_n(u_0, u_1, \dots, u_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} N \left(\sum_{k=0}^n u_k e^{ik\lambda} \right) e^{-in\lambda} d\lambda, \quad \forall n \in \mathbb{N}_0. \quad (4.6)$$

Second Method. We can also calculate Adomian polynomials recursively. Define an operator T by

$$T(A_n(u_0, u_1, \dots, u_n)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} A_n(v_0, v_1, \dots, v_n) e^{-i\lambda} d\lambda, \quad (4.7)$$

where $v_k = u_k + (k+1)u_{k+1}e^{i\lambda}$ and in view of Remark 2.1, we put $\bar{v}_k = \bar{u}_k + (k+1)\bar{u}_{k+1}e^{i\lambda}$, $\forall k \in \{0, 1, 2, \dots, n\}$.

Lemma 4.1. Let $u = \sum_{k=0}^{\infty} u_k$ be the solution of (1.1) and N be a nonlinear operator. Then, operator T given by (4.7) satisfies the following properties.

- (i) $T(u_k) = (k+1)u_{k+1}$, $\forall k \in \mathbb{N}_0$,
- (ii) $T(N^{(k)}(u_0)) = u_1 N^{(k+1)}(u_0)$, $\forall k \in \mathbb{N}_0$,
- (iii) $T(u_{k_1} u_{k_2} \dots u_{k_m}) = u_{k_1} T(u_{k_2} u_{k_3} \dots u_{k_m}) + u_{k_2} u_{k_3} \dots u_{k_m} T(u_{k_1})$, $\forall m \in \mathbb{N}$, $m \geq 2$,
- (iv) $T(u_{k_1} \dots u_{k_m} N^{(k)}(u_0)) = u_{k_1} \dots u_{k_m} T(N^{(k)}(u_0)) + T(u_{k_1} \dots u_{k_m}) N^{(k)}(u_0)$, $\forall m \in \mathbb{N}$, $m \geq 2$,
- (v) $T(\alpha u_{k_1} \dots u_{k_m} N^{(k)}(u_0) + \beta u_{j_1} \dots u_{j_l} N^{(k')}(u_0)) = \alpha T(u_{k_1} \dots u_{k_m} N^{(k)}(u_0)) + \beta T(u_{j_1} \dots u_{j_l} N^{(k')}(u_0))$, $\forall m, l \in \mathbb{N}$, $m, l \geq 2$, where α, β are scalars.

Proof. Parts (i), (iii) and (v) follow easily by using (4.1).

(ii) From (4.7), we have

$$T(N^{(k)}(u_0)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} N^{(k)}(u_0 + u_1 e^{i\lambda}) e^{-i\lambda} d\lambda, \quad \forall k \in \mathbb{N}_0. \quad (4.8)$$

From (4.6), lhs of (4.8) is A_1 for $N^{(k)}(u)$, which by (1.7) is equal to $u_1 N^{(k+1)}(u_0)$.

(iv) Using (2.2) and (4.1), we get

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} N^{(k)}(u_0 + u_1 e^{i\lambda}) e^{-in\lambda} d\lambda = 0, \quad \forall n \in \mathbb{N}. \quad (4.9)$$

From (4.7) and (4.9),

$$\begin{aligned} & T(u_{k_1} \dots u_{k_m} N^{(k)}(u_0)) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \prod_{j=1}^m (u_{k_j} + (k_j + 1)u_{k_j+1} e^{i\lambda}) N^{(k)}(u_0 + u_1 e^{i\lambda}) e^{-i\lambda} d\lambda \\ &= \prod_{j=1}^m u_{k_j} \frac{1}{2\pi} \int_{-\pi}^{\pi} N^{(k)}(u_0 + u_1 e^{i\lambda}) e^{-i\lambda} d\lambda \\ &\quad + \sum_{l=1}^m (k_l + 1) u_{k_l+1} \prod_{\substack{j=1 \\ j \neq l}}^m u_{k_j} \frac{1}{2\pi} \int_{-\pi}^{\pi} N^{(k)}(u_0 + u_1 e^{i\lambda}) d\lambda \\ &= u_{k_1} \dots u_{k_m} T(N^{(k)}(u_0)) + T(u_{k_1} \dots u_{k_m}) N^{(k)}(u_0) \quad \forall m \geq 2. \end{aligned}$$

The last equality follows from (4.8) and Theorem 4.1. \square

For an operator T satisfying above properties, the following result due to Babolian and Javadi [5] holds:

$$A_n(u_0, u_1, \dots, u_n) = \frac{1}{n} T(A_{n-1}(u_0, u_1, \dots, u_{n-1})). \quad (4.10)$$

After calculating A_0 from (4.6) as

$$A_0(u_0) = N(u_0), \quad (4.11)$$

A_n can be calculated by the following recursive formula, obtained using (4.7) and (4.10),

$$A_n(u_0, u_1, \dots, u_n) = \frac{1}{2n\pi} \int_{-\pi}^{\pi} A_{n-1}(v_0, v_1, \dots, v_{n-1}) e^{-i\lambda} d\lambda, \quad \forall n \in \mathbb{N}, \quad (4.12)$$

where $v_k = u_k + (k+1)u_{k+1}e^{i\lambda}$ and $\bar{v}_k = \bar{u}_k + (k+1)\bar{u}_{k+1}e^{i\lambda}$, $\forall k \in \{0, 1, 2, \dots, n-1\}$.

5. NEW METHODS APPLIED TO DIFFERENT FORMS OF NONLINEARITY

We will frequently apply the first method to calculate Adomian polynomials as it is much simpler. By using proposed methods and properties of Adomian polynomials, A_n can be determined easily without manipulations of indices, rearrangement of infinite series and any calculations of derivatives. The second method is efficient in cases where Taylor series expansion is required, as for example in case of exponential, logarithmic and trigonometric nonlinearity. The advantage is that second algorithm requires at most the first two terms of the Taylor series expansion. Applications of properties of Adomian polynomials discussed in third section are also illustrated.

5.1. Nonlinear polynomials.

Example 5.1. (By First Algorithm) Adomian polynomials for $N(u) = u^m$, where $m \in \mathbb{N}$.

We use (4.6) to find A_n . Obviously, $A_0 = u_0^m$ and

$$\begin{aligned}
A_1 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (u_0 + u_1 e^{i\lambda})^m e^{-i\lambda} d\lambda \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\sum_{k=0}^m \binom{m}{k} u_0^k (u_1 e^{i\lambda})^{m-k} \right] e^{-i\lambda} d\lambda \\
&= m u_0^{m-1} u_1, \\
A_2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (u_0 + u_1 e^{i\lambda} + u_2 e^{2i\lambda})^m e^{-2i\lambda} d\lambda \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\sum_{\substack{k_1, k_2, k_3 \\ \sum_{j=1}^3 k_j = m}} \binom{m}{k_1, k_2, k_3} u_0^{k_1} (u_1 e^{i\lambda})^{k_2} (u_2 e^{2i\lambda})^{k_3} \right] e^{-2i\lambda} d\lambda \\
&= m u_0^{m-1} u_2 + \frac{1}{2} m(m-1) u_0^{m-2} u_1^2.
\end{aligned}$$

Similarly A_3, A_4, \dots can be calculated. Indeed, from Theorem 3.1 (ii), the n -th order Adomian polynomial is given by

$$A_n(u_0, u_1, \dots, u_n) = \sum_{\substack{\sum_{j=1}^m k_j = n \\ k_j \in \mathbb{N}_0}} \prod_{j=1}^m u_{k_j}, \quad \forall n \in \mathbb{N}_0. \quad (5.1)$$

Note that from Theorem 3.1 (iv), $A_0 = u_0^m$ and $A_n = \frac{1}{n u_0} \sum_{k=1}^n (k m - n + k) u_k A_{n-k}$.

Remark 5.1. The case $m = 2$ with $N(u) = u^2$ appears in nonlinear fractional wave equation

$$(D_t^{\frac{3}{2}} - D_t^{\frac{1}{2}})u + u_{xx} + u^2 = 0, \quad u(x, 0) = x, \quad u_t(x, 0) = \sin x, \quad t > 0. \quad (5.2)$$

Here, D_t^α is the Caputo fractional derivative of order α , defined by

$$D_t^\alpha u(x, t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m u(x, \tau)}{\partial \tau^m} d\tau, & \text{for } m-1 < \alpha < m, \\ \frac{\partial^m u(x, t)}{\partial t^m}, & \text{for } \alpha = m \in \mathbb{N}. \end{cases} \quad (5.3)$$

Gejji and Bhalekar [9] used existing method to calculate Adomian polynomials. From (5.1), Adomian polynomials for u^2 are

$$A_n(u_0, u_1, \dots, u_n) = \sum_{k=0}^n u_k u_{n-k}, \quad \forall n \in \mathbb{N}_0, \quad (5.4)$$

or recursively, $A_0 = u_0^2$ and $A_n = \frac{1}{n u_0} \sum_{k=1}^n (3k - n) u_k A_{n-k}$, $\forall n \in \mathbb{N}$.

5.2. Nonlinear fractional derivatives.

Example 5.2. Let $m \in \mathbb{N}$ and consider $N(u) = u^m L(u)$, where L is fractional differential or integral operator given by (5.3) or (7.3) respectively. For such operators the Adomian polynomials are simply $L(u_n)$. By using Theorem 3.1 (ii), we get

$$\begin{aligned} A_n(u_0, u_1, \dots, u_n) &= \sum_{k=0}^n B_k L(u_{n-k}) \\ &= \sum_{\substack{\sum_{j=1}^{m+1} k_j = n \\ k_j \in \mathbb{N}_0}} \prod_{j=1}^m u_{k_j} L(u_{k_{m+1}}), \quad \forall n \in \mathbb{N}_0, \end{aligned} \quad (5.5)$$

where B_k 's are the Adomian polynomials for u^m given by (5.1).

Remark 5.2. When $m = 2$ and $L(u) = \frac{\partial u}{\partial x}$, the nonlinear term in

$$u_t + u^2 \frac{\partial u}{\partial x} = 0, \quad u(x, 0) = 3x, \quad (5.6)$$

is $N(u) = u^2 \frac{\partial u}{\partial x}$. Wazwaz [15] used sum of the indices technique and later Biazar *et al.* [6] calculated the Adomian polynomials using different approaches. From (5.5), the n -th order Adomian polynomial is

$$A_n(u_0, u_1, \dots, u_n) = \sum_{\substack{\sum_{j=1}^3 k_j = n \\ k_j \in \mathbb{N}_0}} u_{k_1} u_{k_2} \frac{\partial u_{k_3}}{\partial x}, \quad \forall n \in \mathbb{N}_0. \quad (5.7)$$

The case $m = 1$ with $L(u) = D_x^\beta u$, and $N(u) = u D_x^\beta u$ appears in time and space fractional nonlinear Burger's equation,

$$D_t^\alpha u = v D_x^\beta (D_x^\beta u) - \lambda u D_x^\beta u, \quad u(x, 0) = x^2, \quad t > 0, \quad 0 < \alpha, \beta \leq 1. \quad (5.8)$$

Gepreel [10] used existing techniques to compute the Adomian polynomials for $N(u) = u D_x^\beta u$, where $D_x^\alpha u$ is given by (5.3). From (5.5), we obtain

$$A_n(u_0, u_1, \dots, u_n) = \sum_{k=0}^n u_k D_x^\beta u_{n-k}, \quad \forall n \in \mathbb{N}_0. \quad (5.9)$$

5.3. Trigonometric and hyperbolic functions.

Example 5.3. (By First Algorithm) Adomian polynomials for $N(u) = \cosh u + \sin u$.

Using (4.6), $A_0 = \cosh u_0 + \sin u_0$, and

$$\begin{aligned} A_1 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} [\cosh(u_0 + u_1 e^{i\lambda}) + \sin(u_0 + u_1 e^{i\lambda})] e^{-i\lambda} d\lambda \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} [\cosh u_1 e^{i\lambda} \cosh u_0 + \sinh u_1 e^{i\lambda} \sinh u_0 \\ &\quad + \cos u_1 e^{i\lambda} \sin u_0 + \sin u_1 e^{i\lambda} \cos u_0] e^{-i\lambda} d\lambda \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\left\{ 1 + \frac{u_1^2 e^{2i\lambda}}{2!} + \dots \right\} \cosh u_0 + \left\{ u_1 e^{i\lambda} + \frac{u_1^3 e^{3i\lambda}}{3!} + \dots \right\} \sinh u_0 \right. \end{aligned}$$

$$\begin{aligned}
& + \left\{ 1 - \frac{u_1^2 e^{2i\lambda}}{2!} + \dots \right\} \sin u_0 + \left\{ u_1 e^{i\lambda} - \frac{u_1^3 e^{3i\lambda}}{3!} + \dots \right\} \cos u_0 \Big] e^{-i\lambda} d\lambda \\
& = u_1 (\cos u_0 + \sinh u_0), \\
A_2 & = \frac{1}{2\pi} \int_{-\pi}^{\pi} [\cosh(u_0 + u_1 e^{i\lambda} + u_2 e^{2i\lambda}) + \sin(u_0 + u_1 e^{i\lambda} + u_2 e^{2i\lambda})] e^{-2i\lambda} d\lambda \\
& = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\left\{ 1 + \frac{(u_1 e^{i\lambda} + u_2 e^{2i\lambda})^2}{2!} + \dots \right\} \cosh u_0 + \{(u_1 e^{i\lambda} + u_2 e^{2i\lambda}) + \dots\} \sinh u_0 \right. \\
& \quad \left. + \left\{ 1 - \frac{(u_1 e^{i\lambda} + u_2 e^{2i\lambda})^2}{2!} + \dots \right\} \sin u_0 + \{(u_1 e^{i\lambda} + u_2 e^{2i\lambda}) - \dots\} \cos u_0 \right] e^{-2i\lambda} d\lambda \\
& = \frac{1}{2} u_1^2 (\cosh u_0 - \sin u_0) + u_2 (\cos u_0 + \sinh u_0), \\
A_3 & = \frac{1}{2\pi} \int_{-\pi}^{\pi} [\cosh(u_0 + u_1 e^{i\lambda} + u_2 e^{2i\lambda} + u_3 e^{3i\lambda}) \\
& \quad + \sin(u_0 + u_1 e^{i\lambda} + u_2 e^{2i\lambda} + u_3 e^{3i\lambda})] e^{-3i\lambda} d\lambda \\
& = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\left\{ 1 + \frac{(u_1 e^{i\lambda} + u_2 e^{2i\lambda} + u_3 e^{3i\lambda})^2}{2!} + \dots \right\} \cosh u_0 \right. \\
& \quad + \left\{ (u_1 e^{i\lambda} + u_2 e^{2i\lambda} + u_3 e^{3i\lambda}) + \frac{(u_1 e^{i\lambda} + u_2 e^{2i\lambda} + u_3 e^{3i\lambda})^3}{3!} + \dots \right\} \sinh u_0 \\
& \quad + \left\{ 1 - \frac{(u_1 e^{i\lambda} + u_2 e^{2i\lambda} + u_3 e^{3i\lambda})^2}{2!} + \dots \right\} \sin u_0 \\
& \quad \left. + \left\{ (u_1 e^{i\lambda} + u_2 e^{2i\lambda} + u_3 e^{3i\lambda}) - \frac{(u_1 e^{i\lambda} + u_2 e^{2i\lambda} + u_3 e^{3i\lambda})^3}{3!} + \dots \right\} \cos u_0 \right] e^{-3i\lambda} d\lambda \\
& = \frac{1}{6} u_1^3 (\sinh u_0 - \cos u_0) + u_3 (\sinh u_0 + \cos u_0) + u_1 u_2 (\cosh u_0 - \sin u_0).
\end{aligned}$$

Similarly, A_4, A_5, \dots can be calculated.

Example 5.4. Adomian polynomials for $N(u) = u^2(\cosh u + \sin u)$.

Using Theorem 3.1 (ii), the n -th order Adomian polynomial for product of two nonlinear operators is

$$A_n(u_0, u_1, \dots, u_n) = \sum_{k=0}^n B_k C_{n-k}, \quad (5.10)$$

where B_n and C_n are Adomian polynomials for u^2 and $(\cosh u + \sin u)$ respectively. From (5.4) and Example 5.3, Adomian polynomials for $u^2(\cosh u + \sin u)$ are

$$\begin{aligned}
A_0 &= u_0^2 (\cosh u_0 + \sin u_0), \\
A_1 &= u_0^2 u_1 (\sinh u_0 + \cos u_0) + 2u_0 u_1 (\cosh u_0 + \sin u_0), \\
A_2 &= (u_0^2 u_2 + 2u_0 u_1^2) (\sinh u_0 + \cos u_0) \\
&\quad + (u_1^2 + 2u_0 u_2) (\cosh u_0 + \sin u_0) + \frac{1}{2} u_0^2 u_1^2 (\cosh u_0 - \sin u_0),
\end{aligned}$$

and etc.

5.4. Exponential and logarithmic functions.

Example 5.5. (By First Algorithm) Adomian polynomials for $N(u) = e^u$.

From (4.6), we have $A_0 = e^{u_0}$ and

$$\begin{aligned}
A_1 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{u_0+u_1 e^{i\lambda}} e^{-i\lambda} d\lambda \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{u_0} \left\{ 1 + \frac{u_1 e^{i\lambda}}{1!} + \dots \right\} e^{-i\lambda} d\lambda \\
&= u_1 e^{u_0}, \\
A_2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{u_0+u_1 e^{i\lambda}+u_2 e^{2i\lambda}} e^{-2i\lambda} d\lambda \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{u_0} \left\{ 1 + \frac{(u_1 e^{i\lambda} + u_2 e^{2i\lambda})}{1!} + \frac{(u_1 e^{i\lambda} + u_2 e^{2i\lambda})^2}{2!} + \dots \right\} e^{-2i\lambda} d\lambda \\
&= \left(u_2 + \frac{u_1^2}{2} \right) e^{u_0},
\end{aligned}$$

and etc. Indeed, from Theorem 3.1 (v), the n -th order Adomian polynomial for e^u is

$$A_n(u_0, u_1, \dots, u_n) = e^{u_0} \sum_{\substack{\sum_{j=1}^n j k_j = n \\ k_j \in \mathbb{N}_0}} \prod_{j=1}^n \frac{u_j^{k_j}}{k_j!}, \quad \forall n \in \mathbb{N}. \quad (5.11)$$

Example 5.6. (By Second Algorithm) Adomian polynomials for $N(u) = \ln u$.

Obviously, $A_0 = \ln u_0$, from (4.11). Also, from (4.12),

$$\begin{aligned}
A_1 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\ln u_0 + u_1 e^{i\lambda}) e^{-i\lambda} d\lambda \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln \left[u_0 \left(1 + \frac{u_1 e^{i\lambda}}{u_0} \right) \right] e^{-i\lambda} d\lambda \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\ln u_0 + \left\{ \frac{u_1 e^{i\lambda}}{u_0} + \dots \right\} \right] e^{-i\lambda} d\lambda \\
&= \frac{u_1}{u_0}, \\
A_2 &= \frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{(u_1 + 2u_2 e^{i\lambda})}{(u_0 + u_1 e^{i\lambda})} e^{-i\lambda} d\lambda \\
&= \frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{(u_1 + 2u_2 e^{i\lambda})}{u_0} \left(1 + \frac{u_1 e^{i\lambda}}{u_0} \right)^{-1} e^{-i\lambda} d\lambda \\
&= \frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{(u_1 + 2u_2 e^{i\lambda})}{u_0} \left\{ 1 - \frac{u_1 e^{i\lambda}}{u_0} + \dots \right\} e^{-i\lambda} d\lambda \\
&= \frac{u_2}{u_0} - \frac{u_1^2}{2u_0^2},
\end{aligned}$$

and etc. Indeed, from Theorem 3.1 (v), we get

$$A_n(u_0, u_1, \dots, u_n) = \sum_{\substack{\sum_{j=1}^n j k_j = n \\ k_j \in \mathbb{N}_0}} \frac{(-1)^{\sum_{j=1}^n k_j - 1} \left(\sum_{j=1}^n k_j - 1 \right)!}{u_0^{\sum_{j=1}^n k_j}} \prod_{j=1}^n \frac{u_j^{k_j}}{k_j!}, \quad \forall n \in \mathbb{N}.$$

5.5. Composite nonlinearity.

Example 5.7. (By First Algorithm) Adomian polynomials for $N(u) = e^{\sin u}$.

Using (4.6) $A_0 = e^{\sin u_0}$, and

$$\begin{aligned} A_1 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\sin(u_0 + u_1 e^{i\lambda})} e^{-i\lambda} d\lambda \\ &= \frac{e^{\sin u_0}}{2\pi} \int_{-\pi}^{\pi} e^{\left(\sin u_1 e^{i\lambda} \cos u_0 - 2 \sin^2 \frac{u_1 e^{i\lambda}}{2} \sin u_0 \right)} e^{-i\lambda} d\lambda \\ &= \frac{e^{\sin u_0}}{2\pi} \int_{-\pi}^{\pi} \left[1 + \left(\{u_1 e^{i\lambda} - \dots\} \cos u_0 - 2 \left\{ \frac{1}{2} u_1 e^{i\lambda} - \dots \right\}^2 \sin u_0 \right) + \dots \right] e^{-i\lambda} d\lambda \\ &= u_1 \cos u_0 e^{\sin u_0}, \\ A_2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\sin(u_0 + u_1 e^{i\lambda} + u_2 e^{2i\lambda})} e^{-2i\lambda} d\lambda \\ &= \frac{e^{\sin u_0}}{2\pi} \int_{-\pi}^{\pi} \left[1 + \left(\{(u_1 e^{i\lambda} + u_2 e^{2i\lambda}) - \dots\} \cos u_0 - 2 \left\{ \frac{1}{2} (u_1 e^{i\lambda} + u_2 e^{2i\lambda}) - \dots \right\}^2 \sin u_0 \right) \right. \\ &\quad \left. + \frac{1}{2!} \left(\{(u_1 e^{i\lambda} + u_2 e^{2i\lambda}) - \dots\} \cos u_0 - 2 \left\{ \frac{1}{2} (u_1 e^{i\lambda} + u_2 e^{2i\lambda}) - \dots \right\}^2 \sin u_0 \right)^2 + \dots \right] e^{-2i\lambda} d\lambda \\ &= \left(u_2 \cos u_0 - \frac{1}{2} u_1^2 \sin u_0 + \frac{1}{2} u_1^2 \cos^2 u_0 \right) e^{\sin u_0}. \end{aligned}$$

Using Theorem 3.1 (v), we obtain

$$A_n(u_0, u_1, \dots, u_n) = e^{\sin u_0} \sum_{\substack{\sum_{j=1}^n j k_j = n \\ k_j \in \mathbb{N}_0}} \prod_{j=1}^n \frac{B_j^{k_j}}{k_j!}, \quad \forall n \in \mathbb{N}, \quad (5.12)$$

where B_n are Adomian polynomials of $\sin u$, calculated in Example 5.3.

Example 5.8. Adomian polynomials for $N(u) = e^{-\sin^2 \frac{u}{2}}$.

Adomian and Rach [4] calculated Adomian polynomials for this nonlinear term and later Zhu *et al.* [16] used their algorithm.

Note that $N(u) = e^{-\sin^2 \frac{u}{2}} = e^{-\frac{1}{2}} e^{\frac{1}{2} \cos u_0}$. From Theorem 3.1 (v), we get

$$A_n(u_0, u_1, \dots, u_n) = e^{-\sin^2 \frac{u_0}{2}} \sum_{\substack{\sum_{j=1}^n j k_j = n \\ k_j \in \mathbb{N}_0}} \prod_{j=1}^n \frac{B_j^{k_j}}{k_j!}, \quad \forall n \in \mathbb{N}, \quad (5.13)$$

where B_n are Adomian polynomials for $\frac{1}{2} \cos u$, which can be easily calculated by our first method. Using (4.6), we have $A_0 = e^{-\sin^2 \frac{u_0}{2}}$ and from (5.13),

$$\begin{aligned} A_1 &= -\frac{u_1}{2} \sin u_0 e^{-\sin^2 \frac{u_0}{2}}, \\ A_2 &= \left(-\frac{u_2}{2} \sin u_0 + \frac{u_1^2}{8} \sin^2 u_0 - \frac{u_1^2}{4} \cos u_0 \right) e^{-\sin^2 \frac{u_0}{2}}, \\ A_3 &= \left(-\frac{u_3}{2} \sin u_0 + \frac{u_1^3}{12} \sin u_0 + \frac{u_1 u_2}{4} \sin^2 u_0 \right. \\ &\quad \left. - \frac{u_1 u_2}{2} \cos u_0 + \frac{u_1^3}{16} \sin 2u_0 - \frac{u_1^3}{48} \sin^3 u_0 \right) e^{-\sin^2 \frac{u_0}{2}}, \end{aligned}$$

and etc.

6. EXTENSION TO NONLINEARITY OF SEVERAL VARIABLES

Our methods can be extended to calculate Adomian polynomials for multivariable case also. Consider the system of m functional equations,

$$u_j = f_j + L_j(u_1, u_2, \dots, u_m) + N_j(u_1, u_2, \dots, u_m), \quad j = 1, 2, \dots, m. \quad (6.1)$$

Here L_j and N_j are linear and nonlinear operators respectively and f_j are known functions. As assumed earlier, we shall suppose

H3 : Solution $u_j = \sum_{k=0}^{\infty} u_{j_k}$ of (6.1) are absolutely convergent for $j = 1, 2, \dots, m$.

H4 : The nonlinear function $N_j(u_1, u_2, \dots, u_m)$ is developable into an entire series with infinite radius of convergence, so that for each $j = 1, 2, \dots, m$ we have,

$$N_j(u_1, u_2, \dots, u_m) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \dots \sum_{k_m=0}^{\infty} \frac{\partial^{k_1+\dots+k_m} N_j(0, 0, \dots, 0)}{\partial^{k_1} u_1 \dots \partial^{k_m} u_m} \prod_{j=1}^m \frac{u_j^{k_j}}{k_j!}. \quad (6.2)$$

Since (6.2) is absolutely convergent, it can be rearranged as

$$\begin{aligned} N_j(u_1, u_2, \dots, u_m) &= \sum_{k=0}^{\infty} A_{j_k}(u_{1_0}, \dots, u_{1_k}, u_{2_0}, \dots, u_{2_k}, \dots, u_{m_0}, \dots, u_{m_k}) \\ &= \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \dots \sum_{k_m=0}^{\infty} \frac{\partial^{k_1+\dots+k_m} N_j(u_0, \dots, u_0)}{\partial^{k_1} u_1 \dots \partial^{k_m} u_m} \prod_{j=1}^m \frac{(u_j - u_0)^{k_j}}{k_j!}. \end{aligned} \quad (6.3)$$

Note that (6.3) is a rearrangement of an absolutely convergent series (6.2).

Parameterize $u_j(x, t)$ and its complex conjugate $\bar{u}_j(x, t)$ as follows:

$$u_{j\lambda} = \sum_{k=0}^{\infty} u_{j_k} f^k(\lambda) \text{ and } \bar{u}_{j\lambda}(x, t) = \sum_{k=0}^{\infty} \bar{u}_{j_k} f^k(\lambda), \quad \forall \quad j = 1, 2, \dots, m, \quad (6.4)$$

where λ is a real parameter, f is any real or complex valued function with $|f| < 1$.

Since series (6.3) is absolutely convergent, $N_j(u_{1\lambda}, u_{2\lambda}, \dots, u_{m\lambda})$ can be decomposed as

$$N_j(u_{1\lambda}, u_{2\lambda}, \dots, u_{m\lambda}) = \sum_{k=0}^{\infty} A_{j_k}(u_{1_0}, \dots, u_{1_k}, u_{2_0}, \dots, u_{2_k}, \dots, u_{m_0}, \dots, u_{m_k}) f^k(\lambda), \quad (6.5)$$

for $j = 1, 2, \dots, m$. Taking $f(\lambda) = e^{i\lambda}$, the parametrized form of $u_j(x, t)$, for each j , is given by

$$u_{j\lambda} = \sum_{k=0}^{\infty} u_{j_k} e^{ik\lambda} \quad (6.6)$$

and its complex conjugates, $\bar{u}_j(x, t)$ is parametrized as $\bar{u}_{j\lambda} = \sum_{k=0}^{\infty} \bar{u}_{j_k} e^{ik\lambda}$. We first give the extended version of Theorem 4.1 for the multivariable case.

Theorem 6.1. Let the parametrized representation of $u_j(x, t)$ for $j = 1, 2, \dots, m$ be given by (6.6), where λ is a real parameter and $N_j(u_1, u_2, \dots, u_m)$ are the nonlinear terms in (6.1). Then

$$\int_{-\pi}^{\pi} N_j(u_{1\lambda}, u_{2\lambda}, \dots, u_{m\lambda}) e^{-in\lambda} d\lambda = \int_{-\pi}^{\pi} N_j \left(\sum_{k=0}^n u_{1_k} e^{ik\lambda}, \dots, \sum_{k=0}^n u_{m_k} e^{ik\lambda} \right) e^{-in\lambda} d\lambda. \quad (6.7)$$

Proof. For convenience, we use m dimensional multi-index m -tuple notation.

Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$, $\mathbf{u} = (u_1, u_2, \dots, u_m)$, $\mathbf{u}_\lambda = (\sum_{k=0}^{\infty} u_{1_k} e^{ik\lambda}, \dots, \sum_{k=0}^{\infty} u_{m_k} e^{ik\lambda})$, $\mathbf{u}_{n\lambda} = (\sum_{k=0}^n u_{1_k} e^{ik\lambda}, \dots, \sum_{k=0}^n u_{m_k} e^{ik\lambda})$, and $\mathbf{u}_0 = (u_0, u_0, \dots, u_0)$. Then $|\alpha| = \sum_{k=1}^m \alpha_k$, $\alpha! = \prod_{k=1}^m \alpha_k!$, $\mathbf{u}^\alpha = \prod_{k=1}^m u_k^{\alpha_k}$ and $\partial^\alpha = \prod_{k=1}^m \frac{\partial^{\alpha_k}}{\partial u_k^{\alpha_k}}$.

From H3, $\sum_{k=0}^{\infty} |u_{j_k}| = M_j < \infty$ for $j = 1, 2, \dots, m$ and therefore

$$\left| \partial^\alpha N_j(\mathbf{u}_0) \frac{(\mathbf{u}_\lambda - \mathbf{u}_0)^\alpha}{\alpha!} \right| \leq \left| \frac{\partial^\alpha N_j(\mathbf{u}_0)}{\alpha!} \right| \mathbf{M}^\alpha, \quad (6.8)$$

where $\mathbf{M} = (M_1, M_2, \dots, M_m)$. Using H4, $\sum_{|\alpha| \geq 0} \left| \frac{\partial^\alpha N_j(\mathbf{u}_0)}{\alpha!} \right| \mathbf{M}^\alpha$ converges. Hence, by Weierstrass M-test,

$$\sum_{|\alpha| \geq 0} \partial^\alpha N_j(\mathbf{u}_0) \frac{(\mathbf{u}_\lambda - \mathbf{u}_0)^\alpha}{\alpha!}$$

converges uniformly. Hence, for $n \in \mathbb{N}_0$ and using (6.3), we get

$$\begin{aligned} \int_{-\pi}^{\pi} N_j(\mathbf{u}_\lambda) e^{-in\lambda} d\lambda &= \int_{-\pi}^{\pi} \sum_{|\alpha| \geq 0} \partial^\alpha N_j(\mathbf{u}_0) \frac{(\mathbf{u}_\lambda - \mathbf{u}_0)^\alpha}{\alpha!} e^{-in\lambda} d\lambda \\ &= \int_{-\pi}^{\pi} \lim_{m \rightarrow \infty} \sum_{|\alpha|=m} \partial^\alpha N_j(\mathbf{u}_0) \frac{(\mathbf{u}_\lambda - \mathbf{u}_0)^\alpha}{\alpha!} e^{-in\lambda} d\lambda \\ &= \lim_{m \rightarrow \infty} \int_{-\pi}^{\pi} \sum_{|\alpha|=m} \partial^\alpha N_j(\mathbf{u}_0) \frac{(\mathbf{u}_\lambda - \mathbf{u}_0)^\alpha}{\alpha!} e^{-in\lambda} d\lambda. \\ &= \lim_{m \rightarrow \infty} \sum_{|\alpha|=m} \int_{-\pi}^{\pi} \partial^\alpha N_j(\mathbf{u}_0) \frac{(\mathbf{u}_{n\lambda} - \mathbf{u}_0)^\alpha}{\alpha!} e^{-in\lambda} d\lambda \quad (\text{using (4.1)}) \\ &= \lim_{m \rightarrow \infty} \int_{-\pi}^{\pi} \sum_{|\alpha|=m} \partial^\alpha N_j(\mathbf{u}_0) \frac{(\mathbf{u}_{n\lambda} - \mathbf{u}_0)^\alpha}{\alpha!} e^{-in\lambda} d\lambda \\ &= \int_{-\pi}^{\pi} \lim_{m \rightarrow \infty} \sum_{|\alpha|=m} \partial^\alpha N_j(\mathbf{u}_0) \frac{(\mathbf{u}_{n\lambda} - \mathbf{u}_0)^\alpha}{\alpha!} e^{-in\lambda} d\lambda \end{aligned}$$

$$\begin{aligned}
&= \int_{-\pi}^{\pi} \sum_{|\alpha| \geq 0} \partial^\alpha N_j(\mathbf{u}_0) \frac{(\mathbf{u}_{n_\lambda} - \mathbf{u}_0)^\alpha}{\alpha!} e^{-in\lambda} d\lambda \\
&= \int_{-\pi}^{\pi} N_j(\mathbf{u}_{n_\lambda}) e^{-in\lambda} d\lambda,
\end{aligned}$$

and thus the proof is complete using (6.3). \square

Extension of the first method. Note that nonlinear terms $N_j(u_{1_\lambda}, u_{2_\lambda}, \dots, u_{m_\lambda})$ decompose as

$$N_j(u_{1_\lambda}, u_{2_\lambda}, \dots, u_{m_\lambda}) = \sum_{k=0}^{\infty} A_{j_k}(u_{1_0}, \dots, u_{1_k}, u_{2_0}, \dots, u_{2_k}, \dots, u_{m_0}, \dots, u_{m_k}) e^{ik\lambda}, \quad (6.9)$$

for $j = 1, 2, \dots, m$. To determine A_{j_n} , multiply $e^{-in\lambda}$ in (6.9) and integrate both sides w.r.t λ from $-\pi$ to π , to get

$$\int_{-\pi}^{\pi} N_j \left(\sum_{k=0}^{\infty} u_{1_k} e^{ik\lambda}, \dots, \sum_{k=0}^{\infty} u_{m_k} e^{ik\lambda} \right) e^{-in\lambda} d\lambda = \int_{-\pi}^{\pi} \sum_{k=0}^{\infty} A_{j_k} e^{ik\lambda} e^{-in\lambda} d\lambda = 2\pi A_{j_n}. \quad (6.10)$$

The last equality in (6.10) follows due to the uniform convergence of $\sum_{k=0}^{\infty} A_{j_k} e^{i(k-n)\lambda}$. Hence,

$$A_{j_n}(u_{1_0}, \dots, u_{1_n}, \dots, u_{m_0}, \dots, u_{m_n}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} N_j \left(\sum_{k=0}^{\infty} u_{1_k} e^{ik\lambda}, \dots, \sum_{k=0}^{\infty} u_{m_k} e^{ik\lambda} \right) e^{-in\lambda} d\lambda. \quad (6.11)$$

Applying Theorem 6.1, we get for $j = 1, 2, \dots, m$ and $n \in \mathbb{N}_0$,

$$A_{j_n}(u_{1_0}, \dots, u_{1_n}, \dots, u_{m_0}, \dots, u_{m_n}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} N_j \left(\sum_{k=0}^n u_{1_k} e^{ik\lambda}, \dots, \sum_{k=0}^n u_{m_k} e^{ik\lambda} \right) e^{-in\lambda} d\lambda. \quad (6.12)$$

Extension of the second method. As seen earlier, the Adomian polynomials can also be calculated recursively. We define an operator T as

$$T(A_{j_n}(u_{1_0}, \dots, u_{1_n}, \dots, u_{m_0}, \dots, u_{m_n})) = \frac{1}{2\pi} \int_{-\pi}^{\pi} A_{j_n}(v_{1_0}, \dots, v_{1_n}, \dots, v_{m_0}, \dots, v_{m_n}) e^{-i\lambda} d\lambda, \quad (6.13)$$

where $v_{j_k} = u_{j_k} + (k+1)u_{j_{k+1}}e^{i\lambda}$, $\forall k \in \{0, 1, 2, \dots, n\}$. From (6.12), we get for $j = 1, 2, \dots, m$,

$$A_{j_0}(u_{1_0}, u_{2_0}, \dots, u_{m_0}) = N_j(u_{1_0}, u_{2_0}, \dots, u_{m_0}). \quad (6.14)$$

Note that operator T defined in (6.13) satisfies all the properties of Lemma 4.1. Therefore, by applying (4.10), we get the following recursive formula for A_{j_n} ($1 \leq j \leq m$, $n \in \mathbb{N}$) as

$$A_{j_n}(u_{1_0}, \dots, u_{1_n}, \dots, u_{m_0}, \dots, u_{m_n}) = \frac{1}{2n\pi} \int_{-\pi}^{\pi} A_{j_{n-1}}(v_{1_0}, \dots, v_{1_{n-1}}, \dots, v_{m_0}, \dots, v_{m_{n-1}}) e^{-i\lambda} d\lambda, \quad (6.15)$$

where $v_{j_k} = u_{j_k} + (k+1)u_{j_{k+1}}e^{i\lambda}$ and $\bar{v}_{j_k} = \bar{u}_{j_k} + (k+1)\bar{u}_{j_{k+1}}e^{i\lambda}$, $\forall k \in \{0, 1, 2, \dots, n-1\}$.

Example 6.1. (Extended First Method) Consider the nonlinear equation

$$N_j(u_1, u_2, u_3) = u_1 \frac{\partial u_j}{\partial x} + u_2 \frac{\partial u_j}{\partial y} + u_3 \frac{\partial u_j}{\partial z} \quad \forall j = 1, 2, 3.$$

These nonlinear terms appear in the Navier Stokes equation for an incompressible fluid flow defined by

$$\frac{\partial V}{\partial t} + (V \cdot \nabla)V = \frac{\eta}{\rho} \Delta v - \frac{1}{\rho} \nabla p. \quad (6.16)$$

Here x, y, z are spatial components and $V = (u_1, u_2, u_3)$ denotes the speed vector. Seng *et al.* [14] computed the Adomian polynomials for the nonlinear term $(V \cdot \nabla)V$ in (6.16). Using our first algorithm, we calculate A_n with a few steps. From (6.12), Adomian polynomials A_{j_n} for $j = 1, 2, 3$ are

$$\begin{aligned} A_{j_0} &= u_{1_0} \frac{\partial u_{j_0}}{\partial x} + u_{2_0} \frac{\partial u_{j_0}}{\partial y} + u_{3_0} \frac{\partial u_{j_0}}{\partial z}, \\ A_{j_1} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[(u_{1_0} + u_{1_1} e^{i\lambda}) \frac{\partial(u_{j_0} + u_{j_1} e^{i\lambda})}{\partial x} + (u_{2_0} + u_{2_1} e^{i\lambda}) \frac{\partial(u_{j_0} + u_{j_1} e^{i\lambda})}{\partial y} \right. \\ &\quad \left. + (u_{3_0} + u_{3_1} e^{i\lambda}) \frac{\partial(u_{j_0} + u_{j_1} e^{i\lambda})}{\partial z} \right] e^{-i\lambda} d\lambda \\ &= u_{1_0} \frac{\partial u_{j_1}}{\partial x} + u_{1_1} \frac{\partial u_{j_0}}{\partial x} + u_{2_0} \frac{\partial u_{j_1}}{\partial y} + u_{2_1} \frac{\partial u_{j_0}}{\partial y} + u_{3_0} \frac{\partial u_{j_1}}{\partial z} + u_{3_1} \frac{\partial u_{j_0}}{\partial z}, \\ A_{j_2} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[(u_{1_0} + u_{1_1} e^{i\lambda} + u_{1_2} e^{2i\lambda}) \frac{\partial(u_{j_0} + u_{j_1} e^{i\lambda} + u_{j_2} e^{2i\lambda})}{\partial x} \right. \\ &\quad \left. + (u_{2_0} + u_{2_1} e^{i\lambda} + u_{2_2} e^{2i\lambda}) \frac{\partial(u_{j_0} + u_{j_1} e^{i\lambda} + u_{j_2} e^{2i\lambda})}{\partial y} \right. \\ &\quad \left. + (u_{3_0} + u_{3_1} e^{i\lambda} + u_{3_2} e^{2i\lambda}) \frac{\partial(u_{j_0} + u_{j_1} e^{i\lambda} + u_{j_2} e^{2i\lambda})}{\partial z} \right] e^{-2i\lambda} d\lambda \\ &= u_{1_0} \frac{\partial u_{j_2}}{\partial x} + u_{1_1} \frac{\partial u_{j_1}}{\partial x} + u_{1_2} \frac{\partial u_{j_0}}{\partial x} + u_{2_0} \frac{\partial u_{j_2}}{\partial y} \\ &\quad + u_{2_1} \frac{\partial u_{j_1}}{\partial y} + u_{2_2} \frac{\partial u_{j_0}}{\partial y} + u_{3_0} \frac{\partial u_{j_2}}{\partial z} + u_{3_1} \frac{\partial u_{j_1}}{\partial z} + u_{3_2} \frac{\partial u_{j_0}}{\partial z}. \end{aligned}$$

Thus, by using the extended first method, the Adomian polynomials are given by

$$A_{j_n}(u_{1_0}, \dots, u_{1_n}, \dots, u_{3_0}, \dots, u_{3_n}) = \sum_{(k,w) \in \{(1,x), (2,y), (3,z)\}} \sum_{\substack{a+b=n \\ a,b \in \mathbb{N}_0}} u_{k_a} \frac{\partial u_{j_b}}{\partial w}, \quad \forall n \in \mathbb{N}_0, \quad j = 1, 2, 3.$$

7. AN APPLICATION TO PDE'S

Adomian polynomials for $N(u) = u^m \bar{u}$, where $m \in \mathbb{N}$, can be calculated easily by using (5.5) as

$$A_n(u_0, u_1, \dots, u_n) = \sum_{\substack{\sum_{j=1}^{m+1} k_j = n \\ k_j \in \mathbb{N}_0}} \prod_{j=1}^m u_{k_j} \bar{u}_{k_{m+1}}, \quad \forall n \in \mathbb{N}_0. \quad (7.1)$$

Note when $m = 2$, the nonlinear term, $u^2 \bar{u}$ appears in time fractional nonlinear Schrödinger equation

$$iD_t^\alpha u(x, t) + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + |u|^2 u = 0, \quad u(x, 0) = e^{ix}, \quad 0 < \alpha \leq 1, \quad t > 0, \quad (7.2)$$

where D_t^α is the Caputo fractional derivative of order α given by (5.3). This equation has been solved using ADM by Rida *et al.* [13]. The initial value problem (IVP) (7.2) is equivalent to the integral equation

$$u(x, t) = e^{ix} + iI_t^\alpha \left(\frac{1}{2} \frac{\partial^2 u}{\partial x^2} \right) + iI_t^\alpha (|u|^2 u),$$

that is,

$$\sum_{k=0}^{\infty} u_k = u_0(x, t) + iI_t^\alpha \left(\sum_{k=0}^{\infty} \frac{1}{2} \frac{\partial^2 u_k}{\partial x^2} \right) + iI_t^\alpha \left(\sum_{k=0}^{\infty} A_k \right),$$

where I_t^α is the Riemann-Liouville fractional integral operator of order $\alpha > 0$ defined by

$$I_t^\alpha u(x, t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} u(x, \tau) d\tau, & \text{for } 0 < \alpha, \\ u(x, t), & \text{for } \alpha = 0. \end{cases} \quad (7.3)$$

From (7.1), the Adomian polynomials for the nonlinear term $N(u) = |u|^2 u = u^2 \bar{u}$ are

$$A_n(u_0, u_1, \dots, u_{n-1}, u_n) = \sum_{\substack{\sum_{j=1}^3 k_j = n \\ k_j \in \mathbb{N}_0}} u_{k_1} u_{k_2} \bar{u}_{k_3}, \quad \forall n \in \mathbb{N}_0.$$

Therefore, the first four Adomian polynomials are

$$\begin{aligned} A_0(u_0) &= |u_0|^2 u_0, \\ A_1(u_0, u_1) &= u_0^2 \bar{u}_1 + 2 |u_0|^2 u_1, \\ A_2(u_0, u_1, u_2) &= u_0^2 \bar{u}_2 + u_1^2 \bar{u}_0 + 2 |u_1|^2 u_0 + 2 |u_0|^2 u_2, \\ A_3(u_0, u_1, u_2, u_3) &= u_0^2 \bar{u}_3 + 2 u_0 u_1 \bar{u}_2 + 2 u_0 u_2 \bar{u}_1 + 2 u_1 u_2 \bar{u}_0 + |u_1|^2 u_1 + 2 |u_0|^2 u_3. \end{aligned}$$

Applying ADM, we get

$$u_n = \frac{e^{ix} (it^\alpha)^n}{2^n \Gamma(n\alpha + 1)}, \quad \forall n \in \mathbb{N}_0.$$

Hence, the solution to (7.2) is

$$u(x, t) = e^{ix} E_\alpha \left(\frac{it^\alpha}{2} \right),$$

where $E_\alpha(x)$ is the Mittag-Leffler function [13] of order α defined by

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}.$$

8. CONCLUSIONS

The Adomian decomposition method is a very powerful method for solving nonlinear functional equations of any kind (algebraic, differential, partial differential, integral, fractional differential etc). The crucial step in the method is the employment of the ‘‘Adomian polynomials’’ which allow the solution to converge for the nonlinear portion of the equation, without linearization, perturbation or discretization. However, calculation of Adomian polynomials is in general very tedious as it requires lots of computations.

In this paper, we have discussed some important properties of Adomian polynomials and have developed two new methods which avoid draggy calculation of higher derivatives involved in prevalent methods. Another advantage is that at every stage we don’t have to

keep track of sum of the indices of components of $u(x, t)$. Also, the second algorithm is efficient in cases where Taylor series expansion is required, as for example in case of exponential, logarithmic and trigonometric nonlinearity, and it just requires the first two terms of the Taylor series expansion. We have illustrated our approach using typical examples.

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